TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

ON MILNOR'S INVARIANT FOR LINKS. II. THE CHEN GROUP

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1. **Introduction.** Let G be a group and $\{G_q\}$ the lower central series of G, i.e. $G_1 = G$ and $G_q = [G_{q-1}, G]$ for $q \geqq 2$. Using the second derived group $G'' = [G', G']$, Chen defines a descending series of normal subgroups $\{G(q)\}$ as follows [1]: $G(1) = G$ and $G(q) = G_q G''$ for $q \geqq 2$. For the sake of convenience, we denote $G(\infty) = \bigcap_{q \geqq 1} G(q)$. Then the quotient group $Q(G; q) = G(q)/G(q+1)$ is abelian. If G is finitely generated, so is $Q(G; q)$. $Q(G; q)$ will be called the *Chen group* of G in this paper.

If G is the group of a (polygonal) link L in the 3-sphere, then $Q(G; q)$ is an invariant of the link type. The objective of this paper is to show that if L consists of two components then $Q(G; q)$ is completely determined by its Alexander polynomial $\Delta(x, y)$. More precisely, a group invariant defined by means of $\Delta(x, y)$ completely determines the Chen group of the link group of L. If L consists of more than two components, $Q(G; q)$ may be determined by the Alexander polynomials of its subsets. In particular, if $\Delta(x, y) = 0$ then $Q(G; q)$ is free abelian and conversely. This is a characterization of a link whose Alexander polynomial vanishes. As another characterization of such a link, we shall prove that $\Delta(x, y) = 0$ iff the longitudes of L lie in $G(\infty)$.

We shall use the following notation.

Let G be a group. $[a, b] = aba^{-1}b^{-1}$, for $a, b \in G$. $[a_1, \ldots, a_q] = [[a_1, \ldots, a_{q-1}], a_q]$ for $a_i \in G$. $\langle a_1, \ldots, a_q \rangle$ denotes the normal closure of $a_1, \ldots, a_q$. ∂ denotes the *free derivative* in a free group ring, d the usual *partial derivative* of a function, and 0 the trivializer. Further let

$$D^k(t_1^{\alpha_1}, \ldots, t_\lambda^{\alpha_\lambda})f = \frac{d^k}{dt_1^{\alpha_1}, \ldots, dt_\lambda^{\alpha_\lambda}} f,$$

where the upper suffix k frequently is omitted unless it causes confusion, and let $D^k(\cdots)^0 f = [D^k(\cdots)f]^0$.

2. **A group invariant.** Let G be a finitely presented group such that

(2.1) G has a presentation $\mathscr{P}$ of the deficiency one, $\mathscr{P}(G) = (x_1, \ldots, x_m : r_1, \ldots, r_{m-1})$,

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Presented to the Society, January 24, 1969 under the title *The Chen's group of a link*; received by the editors March 27, 1969.

and

(2.2) the commutator quotient group G/G' is a free abelian group of rank $n \geqq 1$.

Let F be a free group generated by $x_1, \ldots, x_m$. Choose a free basis $\mathfrak{A}=\{t_1, \ldots, t_n\}$ of $A_n=G/G'$. Let $\Phi\colon F\to G$ and $\Psi\colon G\to A_n$ be natural homomorphisms(1). The Alexander matrix $M(\mathscr{P}, \mathfrak{A})=\|\partial r_i/\partial x_j\|^{\Phi\Psi}$ associated to $\mathscr{P}$ and $\mathfrak{A}$ is an $(m-1)\times m$ matrix over the integral group ring ZA_n. The greatest common divisor $\Delta_{\mathfrak{A}}(t_1, \ldots, t_n)$ of all $(m-1)\times(m-1)$ minors of $M(\mathscr{P}, \mathfrak{A})$ will be called the *Alexander polynomial* of $(\mathscr{P}, \mathfrak{A})$. This is an integer polynomial on $t_1, \ldots, t_n$ with possibly negative exponents and is determined uniquely up to a unit $\pm t_1^{\lambda_1}\ldots t_n^{\lambda_n}$ in ZA_n.

Now $\Delta_{\mathfrak{A}}(t_1, \ldots, t_n)$ depends upon the choice of a basis $\mathfrak{A}$. In other words, $\Delta_{\mathfrak{A}}(t_1, \ldots, t_n)$ itself is not a group invariant. (See an example below.) However, by means of $\Delta_{\mathfrak{A}}(t_1, \ldots, t_n)$, we can define a sequence of numerical group invariants $\{A^{(k)}\}$.

Definition 2.1. Let $A^{(-1)}(\mathfrak{A})=0$ and define $A^{(0)}(\mathfrak{A})=\text{ab}\,[\Delta_{\mathfrak{A}}(t_1, \ldots, t_n)]^0$(2). Inductively, we suppose that $A^{(l)}(\mathfrak{A})$ is defined for $0\leqq l\leqq k-1$. For any sequence $1\leqq i_1\leqq i_2\leqq\cdots\leqq i_k\leqq n$, we set

$$A^{(k)}_{i_1,i_2,\ldots,i_k}(\mathfrak{A}) = \text{ab}\left\{\frac{1}{\alpha_1!\cdots\alpha_n!}\, D^k(t_{i_1}, \ldots, t_{i_k})^0\, \Delta_{\mathfrak{A}}(t_1, \ldots, t_n)\right\},$$

where α_j is the number of occurances of j in the sequence $i_1, \ldots. i_k$. Let d be the g.c.d. of $A^{(0)}(\mathfrak{A}), \ldots, A^{(k-1)}(\mathfrak{A})$. Let $\bar{A}^{(k)}_{i_1,\ldots,i_k}(\mathfrak{A})$ be the smallest nonnegative integer such that $\bar{A}^{(k)}_{i_1,\ldots,i_k}\equiv A^{(k)}_{i_1,\ldots,i_k}$ mod d. Then we define

$$A^{(k)}(\mathfrak{A}) = \text{g.c.d.}\,\{\bar{A}^{(k)}_{i_1,\ldots,i_k}(\mathfrak{A})\},$$

where $i_1, \ldots, i_k$ run over all permutations $1, 2, \ldots, n$, subject to $i_1\leqq i_2\leqq\cdots\leqq i_k$. If all $\bar{A}^{(k)}_{i_1,\ldots,i_k}(\mathfrak{A})$ are zero, we define $A^{(k)}(\mathfrak{A})=0$.

From the definition, the following is evident

(2.3) $A^{(k)}(\mathfrak{A})$ is a nonnegative integer and $A^{(k)}(\mathfrak{A}) = 0$ for a sufficiently large k.

Theorem 2.1. *$\{A^{(k)}(\mathfrak{A})\}$ is a group invariant. That is to say, $\{A^{(k)}(\mathfrak{A})\}$ is independent of the choice of presentation and free basis $\mathfrak{A}$ of G/G'. Thus it may be denoted by $\{A^{(k)}(G)\}$.*

Proof. First we should note that $A^{(k)}(\mathfrak{A})$ does not depend upon a particular choice of the Alexander polynomial.

(1) The same symbols will be used for the natural extensions between integral group rings.

(2) ab means "the absolute value of".

Now, from the invariance of the elementary ideals of the Alexander matrix (Chapter VIII, (4.5) in [2]), we need only show that $\{A^{(k)}(\mathfrak{A})\}$ is independent of the choice of basis of G/G'.

Let $\mathscr{B}=\{s_1,\ldots,s_n\}$ be another basis of G/G'. Then $\mathscr{B}$ is obtained from $\mathfrak{A}$ by a finite sequence of the following four transformations.

$$
(2.4)\quad
\begin{array}{lll}
\text{(i)} & t_1\to t_1^{-1},\ t_i\to t_i & \text{for } i\geqq 2,\\
\text{(ii)} & t_1\to t_1t_2,\ t_i\to t_i & \text{for } i\geqq 2,\\
\text{(iii)} & t_1\to t_2,\ t_2\to t_1,\ t_i\to t_i & \text{for } i\geqq 3,\\
\text{(iv)} & t_1\to t_{i+1} & \text{for } 1\leqq i\leqq n-1,\ t_n\to t_1.
\end{array}
$$

Therefore, it is enough to show that $A^{(k)}(\mathfrak{A})$ is unaltered under each transformation. From the definition, it follows immediately that $A^{(k)}(\mathfrak{A})$ is unaltered under (iii) or (iv).

Suppose that $\mathscr{B}$ is obtained from $\mathfrak{A}$ by (i) or (ii). Then it suffices to show the theorem for $n=2$. Let $\mathscr{B}=\{t_1^{\varepsilon}t_2^{\eta}, t_2\}$, $\varepsilon=\pm 1$, $\eta=0$ or 1. By a remark given at the beginning of this proof, we may assume without loss of generality that $\Delta_{\mathscr{B}}(t_1, t_2)=\Delta_{\mathfrak{A}}(t_1^{\varepsilon}t_2^{\eta}, t_2)$. Since $A^{(0)}(\mathscr{B})=|\Delta_{\mathscr{B}}(1, 1)|=|\Delta_{\mathfrak{A}}(1, 1)|=A^{(0)}(\mathfrak{A})$, we can assume inductively that $A^{(l)}(\mathfrak{A})=A^{(l)}(\mathscr{B})$ for $l\leqq k-1$.

Let $w(\alpha,\beta)$ be a sequence $1, 1,\ldots, 1, 2, 2,\ldots, 2$ of length k, where there are α 1's and β 2's. Now by a substitution $t_1=s_1^{\varepsilon}s_2^{\eta}$ and $t_2=s_2$, $\Delta_{\mathfrak{A}}(t_1, t_2)$ becomes the Alexander polynomial $\Delta_{\mathscr{B}}(s_1, s_2)$ associated to $\mathscr{B}$. Therefore,

$$A^{(k)}_{w(\alpha,\beta)}(\mathscr{B}) = \mathrm{ab}\left\{\frac{1}{\alpha!\beta!}D(s_1^{\alpha}, s_2^{\beta})^0\,\Delta_{\mathscr{B}}(s_1, s_2)\right\}.$$

By means of the chain rule and the usual rule for differentiating product, we obtain

$$
(2.5)\quad
\begin{aligned}
\frac{1}{\alpha!\beta!}D(s_1^{\alpha}, s_2^{\beta})^0\,\Delta_{\mathscr{B}}(s_1, s_2) &= \frac{1}{\alpha!\beta!}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}D(t_1^{\alpha+\gamma}, t_2^{\beta-\gamma})^0\,\Delta_{\mathfrak{A}}(t_1, t_2)\varepsilon^{\alpha}\eta^{\beta}\\
&\quad+\sum_{0\leqq\lambda+\mu<\alpha+\beta} f_{\lambda,\mu}(s_1, s_2)^0 D(t_1^{\lambda}, t_2^{\mu})^0\,\Delta_{\mathfrak{A}}(t_1, t_2),
\end{aligned}
$$

where $f_{\lambda,\mu}(s_1, s_2)$ is a certain polynomial on s_1 and s_2. Since $D(t_1^{\lambda}, t_2^{\mu})^0\,\Delta_{\mathfrak{A}}(t_1, t_2)\equiv 0 \bmod A^{(\lambda+\mu)}(\mathfrak{A})$, it follows from (2.5) that

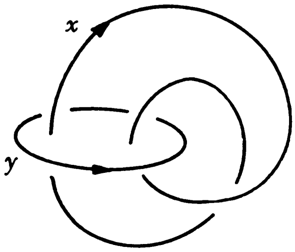
$$(2.6)\quad A^{(k)}_{w(\alpha,\beta)}(\mathscr{B}) \equiv \mathrm{ab}\left\{\sum_{\gamma=0}^{\beta}\varepsilon^{\alpha}\eta^{\gamma}A^{(k)}_{w(\alpha+\gamma,\beta-\gamma)}(\mathfrak{A})\right\}\cdot \bmod \text{g.c.d. }\{A^{(0)}(\mathfrak{A}),\ldots, A^{(k-1)}(\mathfrak{A})\}.$$

This implies that $A^{(k)}(\mathscr{B})=A^{(k)}(\mathfrak{A})$.

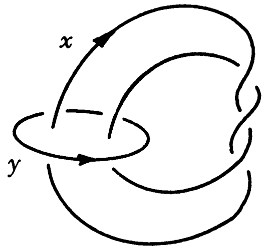
Corollary. *If G/G' is infinite cyclic, then $A^{(0)}(G)=1$ and $A^{(k)}(G)=0$ for $k\geqq 1$.*

If G is the group of a link L, $A^{(k)}(G)$ will be denoted by $A^{(k)}(L)$.

EXAMPLE.



L_1



L_2

The group of two links L_1 and L_2 are isomorphic, since they have the same presentation $(a, b\colon a^2b=ba^2)$. However, the Alexander polynomials are distinct. In fact, $\Delta_1(x, y)=xy+1$ and $\Delta_2(x, y)=x^3y+1$. Then $A^{(0)}(L_1)=A^{(0)}(L_2)=2$, $A^{(1)}(L_1)=A^{(1)}(L_2)=1$ and $A^{(k)}(L_1)=A^{(k)}(L_2)=0$ for $k\geqq 2$. Although L_1 and L_2 are of different isotopy types, their Milnor invariants are coincident: $\mu(12)=2$, $\mu(112)=\mu(122)=1$ and $\mu(i_1i_2\cdots i_n)=0$ for $n\geqq 4$.

3. **Structure of** $F(q)/F(q+1)$. Let F be a free group with m free generators, $x_1,\ldots, x_m, m<\infty$. Let $\mathscr{S}(F)$ denote the set of generators of F, i.e. $\mathscr{S}(F)=\{x_1,\ldots, x_m\}$. We define an order "$<$" in $\mathscr{S}(F)$ in such a way that $x_i<x_j$ iff $i<j$. Now any element of $Q(F; q)$ is a finite product of elements $[a_1,\ldots, a_q]F(q+1)$, $a_i\in F$. An element $[a_1,\ldots, a_q]$ of $F(q)$ is called a *normal element* (of length q) if all a_i lie in $\mathscr{S}(F)$. Further, an element $[a_1,\ldots, a_q]$ is said to be *standard* if it is a normal element and if $a_1<a_2$ and $a_1\leqq a_3\leqq\cdots\leqq a_q$. Any standard element is of the form:

$$[x_i, x_j, \underbrace{x_i,\ldots, x_i}_{\alpha_i \text{ times}}, \underbrace{x_{i+1},\ldots, x_{i+1}}_{\alpha_{i+1} \text{ times}},\ldots, \underbrace{x_m,\ldots, x_m}_{\alpha_m \text{ times}}].$$

For the sake of simplicity this element will be denoted by $[x_i, x_j, x_i^{\alpha_i},\ldots, x_m^{\alpha_m}]$, $\alpha_i,\ldots, \alpha_m$ being nonnegative integers.

Now the group $Q(F; q)$ is completely determined by the following.

THEOREM 3.1 (CHEN). *$Q(F; q)$ is a free abelian group whose basis consists of all standard representatives.*

Theorem 3.1 follows easily from Lemmas 3.1–3.7 below.

LEMMA 3.1. *If $a\equiv b_1^{m_1},\ldots, b_k^{m_k}$ mod $F(r)$, then for $q\geqq 1$*

$$(3.1)\qquad [a, c_1,\ldots, c_q] \equiv \prod_{i=1}^{k} [b_i, c_1,\ldots, c_q]^{m_i} \bmod F(r+q),$$

and for $q\geqq 2$,

$$(3.2)\qquad [c_1,\ldots, c_q, a] \equiv \prod_{i=1}^{k} [c_1,\ldots, c_q, b_i]^{m_i} \bmod F(r+q).$$

LEMMA 3.2. *For* $q \geqq 3$, $[a_1, a_2, a_3, \ldots, a_q] \equiv [a_2, a_1, a_3, \ldots, a_q]^{-1} \bmod F''$.

LEMMA 3.3. $[a_1, a_2, a_3, \ldots, a_q] \equiv [a_1, a_2, a_{\sigma(3)}, \ldots, a_{\sigma(q)}] \bmod F''$, *where* σ *is a permutation of* $3, \ldots, q$.

LEMMA 3.4.

$$[a_1, a_2, a_3, a_4, \ldots, a_q][a_3, a_1, a_2, a_4, \ldots, a_q][a_2, a_3, a_1, a_4, \ldots, a_q] \equiv 1 \bmod F''.$$

Since Lemmas 3.1–3.4 can easily be verified by induction, proofs will be omitted. (Cf. Lemmas A.1, A.3 and Corollary in [1].)

Let f be an element of F. Let $S_r = S_r(\alpha_1, \ldots, \alpha_m)$ be the set of all (proper) shuffle([3]) of α_1 1's, α_2 2's, $\ldots$, α_m m's, where $\alpha_1 + \cdots + \alpha_m = r$. For $\omega = a_1 a_2 \cdots a_r \in S_r(\alpha_1, \ldots, \alpha_m)$, we denote

$$\frac{\partial^r f}{\partial \omega} = \frac{\partial^r f}{\partial x_{a_1} \cdots \partial x_{a_r}} \quad \text{and} \quad D^r(\omega) = D^r(x_{a_1}, \ldots, x_{a_r}).$$

Then the following lemma is easy.

LEMMA 3.5. *If* $f \in F''$, *then for any* ω *of* S_r *and* x_j *in* $\mathscr{S}(F)$,

$$D^r(\omega)^0(\partial f/\partial x_j) = 0.$$

If $f \in F_{q+1}$, *then for any* ω *of* S_r $(\partial^r f/\partial \omega)^0 = 0$, $0 \leqq r \leqq q$.

Further we can prove

LEMMA 3.6.

$$\frac{1}{\alpha_1! \cdots \alpha_m!} D(x_1^{\alpha_1}, \ldots, x_m^{\alpha_m})^0 f^\phi = \sum_\omega \left(\frac{\partial^r f}{\partial \omega}\right)^{0([4])}, \tag{3.3}$$

where ω *runs over all elements in* S_r.

Proof. For $m=1$, the lemma follows from (3.9) in [3]. Thus we may assume inductively that (3.3) holds for $m-1$. In other words,

$$\frac{1}{\alpha_1! \cdots \alpha_{m-1}!} D(x_1^{\alpha_1}, \ldots, x_{m-1}^{\alpha_{m-1}})^0 f^\phi = \sum_\tau \left(\frac{\partial^{r-\alpha_m} f}{\partial \tau}\right),$$

where τ runs over all elements in $S_{r-\alpha_m}(\alpha_1, \ldots, \alpha_{m-1}, 0)$. Since any element of $S_r(\alpha_1, \ldots, \alpha_m)$ is obtained as a generalized shuffle of a certain element of $S_{r-\alpha_m}$ and α_m m's, it follows from Lemma 3.3 in [4] that

$$\begin{aligned} \sum_\omega \left(\frac{\partial^r f}{\partial \omega}\right)^0 &= \left[\sum_\tau \left(\frac{\partial^{r-\alpha_m} f}{\partial \tau}\right)^0\right] \left[\frac{\partial^{\alpha_m} f}{\partial x_m^{\alpha_m}}\right]^0 \\ &= \frac{1}{\alpha_1! \cdots \alpha_{m-1}!} [D(x_1^{\alpha_1}, \ldots, x_{m-1}^{\alpha_{m-1}})^0 f^\phi] \frac{1}{\alpha_m!} [D(x_m^{\alpha_m})^0 f^\phi] \\ &= \frac{1}{\alpha_1! \cdots \alpha_m!} D(x_1^{\alpha_1}, \ldots, x_m^{\alpha_m})^0 f^\phi, \end{aligned}$$

([3]) For the definition, see [4, p. 82].

([4]) ϕ denotes the natural homomorphism from ZF onto $Z(F/F')$.

where ω and τ, respectively, run over all elements in S_r and $S_{r-\alpha_m}$.

From Lemmas 3.5 and 3.6, we see

LEMMA 3.7. *If* $f \in F(q+1)$, *then for any* $\omega \in S_r$ *and* $x_j \in \mathscr{S}(F)$,

$$D^r(\omega)^0(\partial f/\partial x_j)^\Phi = 0, \qquad 0 \leqq r < q.$$

LEMMA 3.8. *Let g be an element of* $F(q)$. *Then*

$$g \equiv \prod [x_i, x_j, x_i^{\alpha_i}, x_{i+1}^{\alpha_{i+1}}, \ldots, x_m^{\alpha_m}]^{\beta(i,j,\alpha_i,\ldots,\alpha_m)} \bmod F(q+1),$$

where $1 \leqq i \leqq j \leqq m$, $\alpha_i + \cdots + \alpha_m = q-2$ *and*

$$\beta(i,j,\alpha_i,\ldots,\alpha_m) = \frac{(-1)^q}{(\alpha_i+1)!\alpha_{i+1}!\cdots\alpha_m!} D^{q-1}(x_i^{\alpha_i+1}, x_{i+1}^{\alpha_{i+1}}, \ldots, x_m^{\alpha_m})^0 \left(\frac{\partial g}{\partial x_j}\right)^\Phi. \tag{3.4}$$

REMARK. This lemma may be regarded as a generalization of (5.5) in [3].

Proof. Let $g = \prod [x_i, x_j, x_i^{\alpha_i}, \ldots, x_m^{\alpha_m}]^\beta \cdot z$, where $z \in F(q+1)$. Since

$$\left(\frac{\partial}{\partial x_j}[x_i, x_j, x_i^{\alpha_i}, \ldots, x_m^{\alpha_m}]^\beta\right) = (-1)(1-x_i)^{\alpha_i+1}(1-x_{i+1})^{\alpha_{i+1}}\cdots(1-x_m)^{\alpha_m}\beta,$$

it follows that

$$\left(\frac{\partial g}{\partial x_j}\right)^\Phi = \sum (-1)(1-x_i)^{\alpha_i+1}(1-x_{i+1})^{\alpha_{i+1}}\cdots(1-x_m)^{\alpha_m}\beta + \left(\frac{\partial z}{\partial x_j}\right)^\Phi. \tag{3.5}$$

Since $D^{q-1}(\cdots)^0(\partial z/\partial x_j)^\Phi = 0$ by Lemma 3.7, we obtain from (3.5) that

$$\frac{1}{(\alpha_i+1)!\alpha_{i+1}!\cdots\alpha_n!} D(x_i^{\alpha_i+1}, \ldots, x_m^{\alpha_m})^0 \left(\frac{\partial g}{\partial x_j}\right)^\Phi = (-1)^q \beta.$$

This completes the proof.

COROLLARY (CHEN). *For* $q \geqq 2$, *the rank of* $Q(F;q)$ *is*

$$(q-1)\binom{m+q-2}{q}.$$

In particular, if $m=2$, *then* $Q(F;q) \cong Z^{q-1}$(5).

4. **Subgroup $H_\Omega(q)$.** Let F be a free group and let Ω be a nonempty subset of $\mathscr{S}(F)$.

DEFINITION 4.1. $H_\Omega(q)$ is the set of all elements w of F such that for any s, $0 \leqq s < q$, and for any $\alpha_i \in \Omega$ and $\beta \in \mathscr{S}(F)$,

$$D(\alpha_1, \ldots, \alpha_s)^0(\partial w/\partial \beta)^\Phi = 0. \tag{4.1}$$

It is well known that $H_\Omega(1) = F'$ for any Ω. Further, from the definition, it follows that if $q < r$ then $H_\Omega(q) \supset H_\Omega(r)$ and if $\Omega' \supset \Omega$ then $H_\Omega(q) \supset H_{\Omega'}(q)$.

(5) G^r denotes the direct product of r copies of G.

LEMMA 4.1. *$H_\Omega(q)$ is a normal subgroup of F, and $H_\Omega(q) \supset F(q+1)$ for any Ω.*

Proof. It is obvious that $H_\Omega(q)$ is a subgroup of F. Let $g \in F$ and $u \in H_\Omega(q)$. Since $u \in F'$, it follows that $u^\phi = 1$, and hence, $[\partial(gug^{-1})/\partial\beta]^\phi = g^\phi(\partial u/\partial\beta)^\phi$. Therefore, for any $s<q$, $D(\alpha_1, \ldots, \alpha_s)^0[\partial(gug^{-1})/\partial\beta]^\phi = 0$. This is a proof of the first proposition. Moreover, since $H_\Omega(q)$ contains both $H_{\mathscr{S}}(q)$ and F'', and since $H_{\mathscr{S}}(q)$ contains F_{q+1} by Lemma 3.5, it follows that $H_\Omega(q) \supset F_{q+1}F'' = F(q+1)$.

For the sake of convenience, we denote $H_\Omega(\infty) = \bigcap_{r \geqq 1} H_\Omega(r)$.

COROLLARY. *$H_\Omega(\infty) \supset F''$ for any Ω.*

LEMMA 4.2. *Let $g = [z_1, \ldots, z_q]$ be a normal element of $F(q)$. If $z_i \notin \Omega$ for some $i \geqq 3$ or if neither z_1 nor z_2 lies in Ω then $g \in H_\Omega(\infty)$.*

Proof. For any $\beta \in \mathscr{S}(F)$, $(\partial g/\partial\beta)^\phi = (1-z_q)\cdots(1-z_3)((\partial/\partial\beta)[z_1, z_2])^\phi$. If $z_i \notin \Omega$ for $i \geqq 3$, then for any $s \geqq 0$, $D(\alpha_1, \ldots, \alpha_s)(\partial g/\partial\beta)^\phi$ contains $1-z_i$ as a common factor. Hence $D(\alpha_1, \ldots, \alpha_s)^0(\partial g/\partial\beta)^\phi = 0$. Suppose that z_1 and z_2 are not in Ω. Since

$$\left(\frac{\partial[z_1, z_2]}{\partial\beta}\right)^\phi = (1-z_2)\left(\frac{\partial z_1}{\partial\beta}\right)^\phi + (z_1-1)\left(\frac{\partial z_2}{\partial\beta}\right)^\phi,$$

$D(\alpha_1, \ldots, \alpha_s)(\partial g/\partial\beta)^\phi$ contains either $1-z_2$ or z_1-1 as a common factor. Hence its trivializer vanishes.

Now we assume that there is defined a mapping τ from $\mathscr{S}(F)$ into F such that

$$g \equiv \tau(g) \mod F(2). \tag{4.2}$$

Let $N(q)$ be the subgroup generated by all normal elements in $F(q)$. We shall extend τ to a mapping from $N(q)$ to $F(q)$.

DEFINITION 4.2. Let $g = [z_1, \ldots, z_q]$ be a normal element of $F(q)$. If all z_i lie in Ω then $\tau(g) = g$. Otherwise let z_i be the first element not in Ω. Then $\tau(g) = [z_1, \ldots, z_{i-1}, \tau(z_i), z_{i+1}, \ldots, z_q]$. τ is extended in the obvious way to a mapping from $N(q)$ to $F(q)$, which will be denoted by the same letter τ. In particular, we define $\tau(1) = 1$.

From (4.2) and Lemma 3.1, it follows

$$\text{For any } g \in N(q), \qquad g \equiv \tau(g) \mod F(q+1). \tag{4.3}$$

Further, since $g^\phi = g^{\tau\phi}$ for any $g \in N(q)$, we see that the following lemma is an easy consequence of Lemma 4.2.

LEMMA 4.3. *Let $g = [z_1, \ldots, z_q]$ be a normal element of $F(q)$. If z_i is not in Ω for $i \geqq 3$ or if neither z_1 nor z_2 lies in Ω, then $\tau(g) \in H_\Omega(\infty)$.*

By means of τ, we shall define a sequence of mappings $\{\tau_n\}$ from F to $F(n)$.

First, we extend the order in $\mathscr{S}(F)$ lexicographically to an order in the set of normal elements of the same length. More precisely, we say $[a_1, \ldots, a_q] < [b_1, \ldots, b_q]$ iff $a_1 < b_1$ or $a_1 = b_1, \ldots, a_m = b_m, a_{m+1} < b_{m+1}, 0 < m < q$.

Using this order, we can write any element of $F(q)/F(q+1)$ as a finite product of standard elements in a unique way (Theorem 3.1). Namely, for any $g \in F(q)$,

$g \equiv \prod_{i=1}^{k} g_i^{\alpha_i} \bmod F(q+1)$, where $g_1 < g_2 < \cdots < g_k$ and α_i is a nonzero integer. This rule ρ_q which to an element g of $F(q)$ associates a finite product of standard elements $\prod_{i=1}^{k} g_i^{\alpha_i}$ will be called the *standardization*. If g is a normal element of $F(q)$, then $g \equiv \rho_q(g) \bmod F''$.

Now we define τ_n inductively as follows:

For any $g \in F$, $\tau_1(g) = g \in F(1) = F$. Suppose that $\tau_n(g)$ is defined for $n \geqq 1$. Then we define $\tau_{n+1}(g) = \tau_n(g)[\tau\rho_n\tau_n(g)]^{-1}$. Since $\tau_n(g) \equiv \tau\rho_n\tau_n(g) \bmod F(n+1)$, it follows that $\tau_{n+1}(g) \in F(n+1)$.

LEMMA 4.4. *If $g \in F(n)$, then $\tau_r(g) = g$ for $r \leqq n$.*

Proof. For $n=1$, Lemma 4.4 is trivial. Assume that Lemma 4.4 holds for $n-1$. Since $g \in F(n)$ and $\tau_{n-1}(g) = g \equiv 1 \bmod F(n)$, it follows that $\tau_n(g) = \tau_{n-1}(g)\tau(1)^{-1} = \tau_{n-1}(g) = g$.

LEMMA 4.5. *If $g \in F(n) \cap H_\Omega(\infty)$ then $\tau_{n+1}(g) \in F(n+1) \cap H_\Omega(\infty)$.*

Proof. Let $\rho_n(g) = \prod_i g_i^{\alpha_i}$. Since $F(n) \ni g$, $\tau_n(g) = g$, and hence, $\tau_{n+1}(g) = g(\tau\rho_n(g))^{-1}$. Since $\tau_{n+1}(g) \in F(n+1)$, we only need to show that $\tau_{n+1}(g) \in H_\Omega(\infty)$, or equivalently, $\tau\rho_n(g) \in H_\Omega(\infty)$. Now, since $g \in H_\Omega(\infty)$, it follows from Lemma 3.8 that g_i cannot be of the form: $[z_1, z_2, z_3, \ldots, z_n]$, where all z_i except possibly z_2 are in Ω. Thus, Lemma 4.4 follows from Lemma 4.3.

5. **Main Lemma.** Let F be a free group with two disjoint nonempty sets of generators $\Omega = \{x_1, \ldots, x_n\}$ and $\Gamma = \{a_1, \ldots, a_m\}$. Let τ be a mapping from $\mathscr{S}(F)$ to F defined as follows:

$$(5.1) \qquad \begin{aligned} \tau(x_i) &= x_i, & 1 \leqq i \leqq n, \quad &\text{and} \\ \tau(a_i) &= f_{a_i}^{-1} a_i, & 1 \leqq i \leqq m, \quad & \end{aligned}$$

where $f_{a_i} \in F(2)$.

We define an order in $\mathscr{S}(F)$ as follows. Elements in Ω (or Γ) are naturally ordered according to their indices, and any element of Ω is *less* than any of Γ. This order can be extended lexicographically to an order in the set of normal elements of the same length. See §4.

Since τ satisfies (4.2), a sequence of mappings $\{\tau_n\}$ is well defined.

Now, using f_{a_i}, we shall define three endomorphisms ν, σ and ϕ_r ($r \geqq 1$) of F as follows:

$$(5.2) \qquad \begin{aligned} &\text{(i)} & \nu(x_i) &= x_i \quad \text{and} \quad \nu(a_i) = f_{a_i}, \\ &\text{(ii)} & \sigma(x_i) &= x_i \quad \text{and} \quad \sigma(a_i) = 1, \\ &\text{(iii)} & \phi_1 &= \sigma \quad \text{and} \quad \phi_r = \sigma\nu^{r-1}, \text{ for } r \geqq 2. \end{aligned}$$

Consider an element $w = [W, x_1]$ in $F(2)$. We can write $w = \bar{w}\rho_2(w)$, where $\bar{w} \in F(3)$ and $\rho_2(w) \in F(2)$. Since $\tau_h(w) = \tau_{h-1}(w)[\tau\rho_{h-1}\tau_{h-1}(w)]^{-1}$ and $\tau_2(w) = w$, it follows by induction that

$$\tau_h(w) = w[\tau\rho_2\tau_2(w)]^{-1}[\tau\rho_3\tau_3(w)]^{-1} \cdots [\tau\rho_{h-1}\tau_{h-1}(w)]^{-1}.$$

We should note that $\tau_h(w)$ is a finite product of the τ-image of standard elements of the length $<h$. For convenience, we write $\tau_h(w)$ in the following form:

$$\tau_h(w) = \bar{w} \prod_{2 \leqq k \leqq h-1} \prod_i g_{k,i}^{\beta(k,i)}, \tag{5.3}$$

where $g_{k,i}$ is a τ-image of a standard element of $F(k)$. Therefore, $g_{k,i}^{\phi}$ is a standard element of $F(k)$.

On the other hand, since $\tau_h(w) \in F(h)$, we see that $\rho_h \tau_h(w)$ is written as

$$\prod_j g_{h,j}^{\beta(h,j)},$$

where $g_{h,j}$ is a standard element of $F(h)$.

The purpose of this section is to determine the exponent $\beta(k, i)$ of $g_{k,i}$ of some particular type. Denote by $\beta(z_1, \ldots, z_r)$ the exponent of the element $[z_1, \ldots, z_r]$ occurring in $\tau_h(w)$ or $\rho_h \tau_h(w)$. Then we shall prove

LEMMA 5.1.

$$\beta(x_i, x_j, x_i^{\alpha_i}, \ldots, x_n^{\alpha_n}) = \frac{(-1)^{h-1}}{\alpha_i! \cdots \alpha_m!} D(x_i^{\alpha_i}, \ldots, x_n^{\alpha_n})^0 \left(\frac{\partial W^{\tilde{\phi}}}{\partial x_j}\right)^{\phi}, \tag{5.4}$$

where $\tilde{\phi} = \phi_{h-1}$ *and* $\sum_i \alpha_i = h-2$.

Now to avoid unnecessary complications, we shall prove Lemma 5.1 for $n=2$, and use x, y instead of x_1, x_2. In fact, this is what we really need in the subsequent sections.

First we shall establish a relation between $\bar{w}$ and W.

LEMMA 5.2. *If* $z \in \mathscr{S}(F)$ *is different from* x, *then for* $\lambda + \mu > 0$,

$$\frac{1}{\lambda!\mu!} D(x^{\lambda}, y^{\mu})^0 \left(\frac{\partial W}{\partial z}\right)^{\phi} = \frac{(-1)}{(\lambda+1)!\mu!} D(x^{\lambda+1}, y^{\mu}) \left(\frac{\partial \bar{w}}{\partial z}\right)^{\phi}.$$

Proof. Since $w = \bar{w}\rho_2(w)$,

$$\left(\frac{\partial w}{\partial z}\right)^{\phi} = \left(\frac{\partial \rho_2(w)}{\partial z}\right)^{\phi} + \left(\frac{\partial \bar{w}}{\partial z}\right)^{\phi}.$$

On the other hand, $(\partial w/\partial z)^{\phi} = (1-x)(\partial W/\partial z)^{\phi}$, because of $w = [W, x]$. Therefore,

$$(1-x)\left(\frac{\partial W}{\partial z}\right)^{\phi} = \left(\frac{\partial \rho_2(w)}{\partial z}\right)^{\phi} + \left(\frac{\partial \bar{w}}{\partial z}\right)^{\phi}. \tag{5.5}$$

Since $(\partial \rho_2(w)/\partial z)^{\phi}$ is a linear polynomial on x, y or a_j, $D(x^{\lambda+1}, y^{\mu})^0(\partial \rho_2(w)/\partial z)^{\phi} = 0$, for $\lambda + \mu \geqq 1$. Thus from (5.5) we obtain

$$\frac{1}{(\lambda+1)!\mu!} \binom{\lambda+1}{1} (-1) D(x^{\lambda}, y^{\mu})^0 \left(\frac{\partial W}{\partial z}\right)^{\phi} = \frac{1}{(\lambda+1)!\mu!} D(x^{\lambda+1}, y^{\mu})^0 \left(\frac{\partial \bar{w}}{\partial z}\right)^{\phi}.$$

This is the required formula.

The next lemma is a recursion formula involving $\beta(z_1,\dots,z_r)$. Let $u_i=\tau(a_i)$.

LEMMA 5.3. *For* $p+q\leqq h-2$, $p\geqq 1$, $p+q\geqq 2$,

$$\beta(x, u_i, x^{p-1}, y^q) = \sum_{r,s,k}\frac{1}{p!q!}r!s!(-1)^{r+s+p+q-1}\beta(x, u_k, x^{r-1}, y^s)\binom{p}{r}\binom{q}{s} \times D(x^{p-r}, y^{q-s})^0\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi}+\frac{(-1)^{p+q}}{(p-1)!q!}D(x^{p-1}, y^q)^0\left(\frac{\partial W}{\partial a_i}\right)^{\phi},$$

where the summation runs over all r, s and k such that $1\leqq r\leqq p$, $0\leqq s\leqq q$, $1\leqq r+s\leqq p+q-1$, $1\leqq k\leqq m$.

Proof. To obtain the required formula, we shall calculate $D(x^p, y^q)^0(\partial\tau h/\partial a_i)^{\phi}$ from (5.3). Since $p+q\geqq 2$, $D(x^p, y^q)^0((\partial/\partial a_i)[z_1, z_2])^{\phi}=0$ for $z_i\in\mathscr{S}(F)$. Also, since $g_{k,j}$ is a τ-image of a standard element of $F(k)$, it follows from Lemma 4.3 that $g_{k,j}=[z_1,\dots,z_k]\in H_{\Omega}(\infty)$ if $z_i=u_l$ for some $i\geqq 3$ or both z_1^{ϕ} and z_2^{ϕ} are in Γ. Therefore, by excluding those factors which are contained in $H_{\Omega}(\infty)$, we obtain:

For $p+q\geqq 2$,

$$D(x^p, y^q)^0\left(\frac{\partial\tau_h(w)}{\partial a_i}\right)^{\phi} = D(x^p, y^q)^0\left\{\sum_{r,s,k,1\leqq r+s\leqq p+q}(-1)\beta(x, u_k, x^{r-1}, y^s) \times(1-x)^r(1-y)^s\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi}+\left(\frac{\partial\overline{w}}{\partial a_i}\right)^{\phi}\right\}. \tag{5.6}$$

Since $(\partial u_k/\partial a_i)^{\phi 0}=\delta_{i,k}$ and $D(x^p, y^q)^0(\partial\tau_h(w)/\partial a_i)=0$ for $p+q\leqq h-2$, because $\tau_h(w)\in F(h)$, we see that if $p+q\geqq 2$,

$$0 = \sum\sum\beta(x, u_k, x^{r-1}, y^s)\binom{p}{r}\binom{q}{s}r!s!(-1)^{r+s-1}D(x^{p-r}, y^{q-s})^0\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi} +(-1)^{p+q-1}p!q!\beta(x, u_i, x^{p-1}, y^q)+D(x^p, y^q)^0\left(\frac{\partial\overline{w}}{\partial a_i}\right).$$

From this formula and Lemma 5.2, Lemma 5.3 follows.

LEMMA 5.4. *For any q*, $2\leqq q\leqq h-2$,

$$\beta(y, u_i, y^{q-1}) = \sum_{1\leqq s\leqq q-1}\sum_k\frac{1}{q!}s!(-1)^{s+q-1}\beta(y, u_k, y^{s-1})\binom{q}{s} \times D(y^{q-s})^0\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi}+\frac{(-1)^q}{q!}D(y^q)^0\left(\frac{\partial\overline{w}}{\partial a_i}\right)^{\phi}. \tag{5.7}$$

Proof. From (5.3), it follows that if $q\geqq 2$,

$$\begin{aligned}0&=D(y^q)^0\left(\frac{\partial\tau_h(w)}{\partial a_i}\right)\\&=D(y^q)^0\left\{(-1)\beta(y, a_i)(1-y)+\sum_k\sum_{1\leqq s\leqq q}\beta(y, u_k, y^{s-1})(1-y)^s(-1)\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi}+\left(\frac{\partial\overline{w}}{\partial a_i}\right)^{\phi}\right\}\\&=\sum_k\sum_{1\leqq s<q}(-1)^{s-1}\beta(y, u_k, y^{s-1})s!\binom{q}{s}D(y^{q-s})^0\left(\frac{\partial u_k}{\partial a_i}\right)^{\phi}\\&\quad+(-1)^{q-1}q!\beta(y, u_k, y^{q-1})+D(y^q)^0\left(\frac{\partial\overline{w}}{\partial a_i}\right)^{\phi}.\end{aligned}$$

Q.E.D.

As a special case of Lemmas 5.3 and 5.4, we have

LEMMA 5.5. $\beta(x, u_i)=\beta(x, a_i)=(-1)(\partial W/\partial a_i)^{\phi 0}$.

Proof. From the expression

$$w = [W, x] = \bar{w}[x, y]^{\beta(x,y)} \prod_i [x, a_i]^{\beta(x,a_i)},$$

it follows that

$$\left(\frac{\partial w}{\partial a_i}\right)^{\phi} = (1-x)\left(\frac{\partial W}{\partial a_i}\right)^{\phi} = \left(\frac{\partial \bar{w}}{\partial a_i}\right)^{\phi} + (x-1)\beta(x, a_i).$$

Thus, $D(x)^0(\partial w/\partial a_i)^{\phi}=(-1)(\partial W/\partial a_i)^{\phi 0}$. On the other hand, $D(x)^0(\partial \bar{w}/\partial a_i)^{\phi}=0$. Hence we have $D(x)^0(\partial W/\partial a_i)^{\phi}=\beta(x, a_i)$. Thus, Lemma 5.5 follows.

Now we are in position to prove Lemma 5.1.

Since Lemma 5.1 is trivial for $h=2$, we assume that $h>2$. Since $\tau_h(w)\equiv \rho_h\tau_h(w) \bmod F(h+1)$, it follows from Lemma 3.7 that

$$D(x^{\lambda+1}, y^{\mu})^0\left(\frac{\partial \tau_h(w)}{\partial y}\right)^{\phi} = D(x^{\lambda+1}, y^{\mu})^0\left(\frac{\partial \rho_h\tau_h(w)}{\partial y}\right)^{\phi} \quad \text{for } \lambda+\mu+2 = h.$$

Thus, Lemma 3.8 shows that

$$\beta(x, y, x^{\lambda}, y^{\mu}) = \frac{(-1)^{\lambda+\mu}}{(\lambda+1)!\mu!} D(x^{\lambda+1}, y^{\mu})^0\left(\frac{\partial \tau_h(w)}{\partial y}\right)^{\phi}, \tag{5.8}$$

where $h=\lambda+\mu+2$. By a direct calculation, the right-hand side of (5.8) becomes

$$\begin{aligned} &\frac{(-1)^{\lambda+\mu}}{(\lambda+1)!\mu!} D(x^{\lambda+1}, y^{\mu})^0\Big\{\sum_i \sum_{p,q} (-1)\beta(x, u_i, x^{p-1}, y^q) \\ &\qquad\qquad \cdot(1-x)^p(1-y)^q\left(\frac{\partial u_i}{\partial y}\right)^{\phi} + \left(\frac{\partial \bar{w}}{\partial y}\right)^{\phi}\Big\} \\ &= \frac{(-1)^{\lambda+\mu}}{(\lambda+1)!\mu!}\Big\{\sum\sum p!q!(-1)^{p+q+1}\binom{\lambda+1}{p}\binom{\mu}{q}\beta(x, u_i, x^{p-1}, y^q) \\ &\qquad\qquad \cdot D(x^{\lambda+1-p}, y^{\mu-q})^0\left(\frac{\partial u_i}{\partial y}\right)^{\phi} + D(x^{\lambda+1}, y^{\mu})^0\left(\frac{\partial \bar{w}}{\partial y}\right)^{\phi}\Big\}, \end{aligned} \tag{5.9}$$

where the summation runs over all p, q and i such that $1\leqq p\leqq \lambda+1$, $0\leqq q\leqq \mu$, $1\leqq p+q\leqq \lambda+\mu$, $1\leqq i\leqq m$.

Now consider the right-hand side of (5.4). By the chain rule, we have

$$\begin{aligned} &(-1)^{\lambda+\mu-1}\frac{1}{\lambda!\mu!} D(x^{\lambda}, y^{\mu})^0\left(\frac{\partial W^{\phi_{h-1}}}{\partial y}\right)^{\phi} \\ &\qquad = \frac{(-1)^{\lambda+\mu-1}}{\lambda!\mu!} D(x^{\lambda}, y^{\mu})^0\Big\{\left(\frac{\partial W}{\partial y}\right)^{\phi} + \sum_i \left(\frac{\partial W}{\partial a_i}\right)^{\phi}\left(\frac{\partial a_i^{\phi_{h-1}}}{\partial y}\right)^{\phi}\Big\}. \end{aligned} \tag{5.10}$$

Then by Lemma 5.2, the first term of the right-hand side of (5.10) coincides with the last term of the right-hand side of (5.9). Thus we only need to show that

$$\frac{1}{(\lambda+1)!\mu!}\sum\sum\binom{\lambda+1}{p}\binom{\mu}{q}p!q!(-1)^{p+q}\beta(x,u_i,x^{p-1},y^q)\cdot D(x^{\lambda+1-p},y^{\mu-q})^0\left(\frac{\partial u_i}{\partial y}\right)^{\phi} = \frac{1}{\lambda!\mu!}D(x^{\lambda},y^{\mu})^0\sum_i\left(\frac{\partial W}{\partial a_i}\right)^{\phi}\left(\frac{\partial a_i^{\phi_{h-1}}}{\partial y}\right)^{\phi}, \tag{5.11}$$

where p, q run over the same range as defined in (5.8).

Now the chain rule shows that

$$\frac{\partial a_i^{\phi_{h-1}}}{\partial y}=\frac{\partial a_i^{\nu}}{\partial y}+\sum_k\frac{\partial a_i^{\nu}}{\partial a_k}\frac{\partial a_k^{\phi_{h-2}}}{\partial y}\quad\text{and}\quad\frac{\partial a_i^{\phi_2}}{\partial y}=\frac{\partial u_i}{\partial y}.$$

Therefore, by induction, we can prove that

$$\frac{\partial a_i^{\phi_{h-1}}}{\partial y}=\sum_{i\leqq j,\dots,k,l\leqq m}\underbrace{\left(\frac{\partial a_i^{\nu}}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^{\nu}}{\partial a_l}\right)}_{\text{at most } h-2 \text{ factors}}\left(\frac{\partial u_l}{\partial y}\right). \tag{5.12}$$

Noting that $(\partial a_i^{\nu}/\partial a_j)^0=0$, we see that

$$D(x^r,y^s)^0\left(\frac{\partial a_i^{\phi_{h-1}}}{\partial y}\right)^{\phi} = \binom{r}{b}\binom{s}{c}\left[D(x^b,y^c)^0\sum\left(\frac{\partial a_i^{\nu}}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^{\nu}}{\partial a_l}\right)\right]D(x^{r-b},y^{s-c})^0\left(\frac{\partial u_l}{\partial y}\right)^{\phi}. \tag{5.13}$$

By substituting (5.7) and (5.13) into (5.11), we obtain an expression involving $\beta(x,u_i,x^{r-1},y^s)$ and $D(x^r,y^s)(\partial u_l/\partial y)^{\phi}$. Compare the term involving $D(x^{\lambda+1-p},y^{\mu-q})\cdot(\partial u_i/\partial y)^{\phi}$. In the left-hand side of (5.11), the coefficient of this term is

$$\frac{1}{(\lambda+1)!\mu!}(-1)^{p+q}\binom{\lambda+1}{p}\binom{\mu}{q}\beta(x,u_i,x^{p-1},y^q). \tag{5.14}$$

On the other hand, in the right-hand side of (5.11), it is

$$\frac{1}{\lambda!\mu!}\binom{\lambda}{p-1}\binom{\mu}{q}D(x^{p-1},y^q)^0\left[\sum\left(\frac{\partial W}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^{\nu}}{\partial a_i}\right)\right]^{\phi}. \tag{5.15}$$

Thus it is enough to show that these expressions coincide. Then, by a direct computation, it reduces to

$$(-1)^{p+q}\beta(x,u_i,x^{p-1},y^q)=\frac{1}{(p-1)!q!}D(x^{p-1},y^q)^0\left[\left(\frac{\partial W}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^{\nu}}{\partial a_i}\right)\right]^{\phi}. \tag{5.16}$$

Now (5.16) will be proved by induction on $p+q$.

In the case where $p+q=1$, i.e. $p=1$ and $q=0$, (5.16) becomes $(-1)\beta(x,u_i)=(\partial W/\partial a_i)^0$. This is true by Lemma 5.5.

Suppose (5.16) holds for $r+s<p+q$. Then

$$(-1)^{p+q}\beta(x, u_i, x^{p-1}, y^q)$$
$$= \sum\sum \frac{1}{p!q!} r!s!(-1)^{r+s-1}\binom{p}{r}\binom{q}{s} \frac{(-1)^{r+s-1}}{(r-1)!s!} D(x^{r-1}, y^s)^0$$
$$\cdot\left[\sum\left(\frac{\partial W}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^v}{\partial a_l}\right)\left(\frac{\partial u_l}{\partial a_t}\right)\right]^\phi D(x^{p-r}, y^{q-s})^0\left(\frac{\partial u_t}{\partial a_i}\right)^\phi$$

$$(5.17) \qquad +\frac{1}{(p-1)!q!} D(x^{p-1}, y^q)^0\left(\frac{\partial W}{\partial a_i}\right)^\phi$$
$$= \frac{1}{(p-1)!q!} D(x^{p-1}, y^q)^0\left(\frac{\partial W}{\partial a_i}\right)^\phi + \sum \frac{1}{p!q!} r!s!\binom{p}{r}\binom{q}{s} \frac{1}{(r-1)!s!} D(x^{r-1}, y^s)^0$$
$$\cdot\left[\left(\frac{\partial W}{\partial a_j}\right)\cdots\left(\frac{\partial a_k^v}{\partial a_l}\right)\left(\frac{\partial u_l}{\partial a_t}\right)\right]^\phi D(x^{p-r}, y^{q-s})^0\left(\frac{\partial u_t}{\partial a_i}\right)^\phi.$$

Since

$$\frac{1}{p!}\frac{1}{q!} r!s!\binom{p}{r}\binom{q}{s} \frac{1}{(r-1)!s!} = \frac{1}{(p-1)!q!}\binom{p-1}{r-1}\binom{q}{s},$$

(5.17) becomes

$$(-1)^{p+q}(x, u_i, x^{p-1}, y^q) = \frac{1}{(p-1)!q!} D(x^{p-1}, y^q)^0\left\{\left(\frac{\partial W}{\partial a_i}\right) + \sum\left(\frac{\partial W}{\partial a_k}\right)\cdots\left(\frac{\partial u_t}{\partial a_i}\right)\right\}^\phi$$
$$= \frac{1}{(p-1)!q!} D(x^{p-1}, y^q)^0 \sum\left\{\left(\frac{\partial W}{\partial a_k}\right)\cdots\left(\frac{\partial u_t}{\partial a_i}\right)\right\}^\phi.$$

This proves Lemma 5.1.

6. **The group of a link.** Let $\mathscr{P}(G)=(x_{i,j}; r_{i,j})$ be a *Wirtinger* presentation of G of a link with n components. The *standard presentation* $\mathscr{P}_s$ of G is a presentation [6]:

$$\mathscr{P}_s(G) = (x_{i,j}:s_{i,j}, 1 \leqq i \leqq n, 1 \leqq j \leqq \lambda_i),$$

where $s_{i,j}=v_{i,j}x_{i,1}v_{i,j}^{-1}x_{i,j+1}^{-1}$.

Let $\tilde{F}=(x_{i,j}, 1\leqq i\leqq n, 1\leqq j\leqq \lambda_i:)$ and $F=(x_i, a_{i,j}, 1\leqq i\leqq n, 2\leqq j\leqq \lambda_i:)$. Then there exists an isomorphism $\chi: \tilde{F}\to F$ defined by

$$(6.1) \qquad \chi(x_{i,1}) = x_i \quad\text{and}\quad \chi(x_{i,j}) = a_{i,j}^{-1}x_i, \qquad j \geqq 2.$$

By means of χ, we can obtain a new presentation $\mathscr{P}_0$ of G.

$$\mathscr{P}_0(G) = (x_i, a_{i,j}:t_{i,j-1}, t_i, 1 \leqq i \leqq n, 2 \leqq j \leqq \lambda_i),$$

where $t_{i,j-1}=\chi(s_{i,j-1})$ and $t_i=\chi(s_{i,\lambda_i})$.

It is evident that $t_{i,j-1}=[x_i, u_{i,j}]^{-1}a_{i,j}$ and $t_i=[\xi_i, x_i]$, where $u_{i,j}=\chi(v_{i,j})$ and $\xi_i=\chi(v_{i,\lambda_i})$.

In order to apply lemmas obtained in the previous sections on the link group, we shall define a mapping τ.

Let $\Omega=\{x_1,\ldots,x_n\}$ and $\Gamma=\{a_{i,j}\}$.

First, we define the order "<" in $\mathscr{S}(F)=\Omega\cup\Gamma$ as follows:

$$(6.2)\qquad \begin{aligned} x_i &< a_{j,k} && \text{for any } i, j, k,\\ x_i &< x_j && \text{iff } i<j, \text{ and}\\ a_{j,k} &< a_{l,m} && \text{iff } j<l \text{ or } j=l,\ k<m. \end{aligned}$$

We extend this order lexicographically to an order in the set of normal elements of the same length. (See §§4 and 5.)

Now, τ will be defined as

$$(6.3)\qquad \tau(a_{i,j}) = t_{i,j} \quad \text{and} \quad \tau(x_i) = x_i.$$

Since τ satisfies (4.2) and (5.1), $\{\tau_n\}$ also are defined. Further, ν, σ and ϕ_r are defined by (5.2). With these endomorphisms and χ, we introduce another endomorphism $\tilde{\nu}$ of $\tilde{F}$ and $\tilde{\sigma}$ as follows:

$$(6.4)\qquad \tilde{\nu} = \chi^{-1}\nu\chi \quad \text{and} \quad \tilde{\sigma} = \sigma\chi.$$

Then the homomorphism θ_p in [6] is exactly $\tilde{\sigma}\tilde{\nu}^{p-1}$, $p\geqq 1$ and

$$(6.5)\qquad \theta_p = \phi_p\chi, \quad \text{for } p \geqq 1.$$

Now, Theorem 3.1 shows that a free generator of $Q(F;q)$ is represented by a standard element of $F(q)$.

DEFINITION 6.1. A standard element $[z_1,\ldots,z_q]$ of $F(q)$ is said to be *substantial* if every z_i lies in Ω. Otherwise, it is *insubstantial.*

Now the longitude l_i of each component L_i of a link L represents an element of the link group G of L. By a suitable choice of the base point of G, we may assume without loss of generality that l_i is represented by $t_i=[\xi_i, x_i]$ in $\mathscr{P}_0(G)$.

Suppose that L has only two components. Then we can prove

THEOREM 6.1.

$$(6.6)\qquad \frac{1}{p!q!}\,D(x_1^p, x_2^q)^0\left(\frac{\partial \xi_2^{\tilde{\phi}}}{\partial x_1}\right)^{\phi} \equiv \frac{(-1)^q}{p!q!}\,D(x_1^p, x_2^q)^0\,\Delta(x_1, x_2) \bmod A^{(p+q-1)}(L),$$

where $\tilde{\phi}=\phi_{p+q+1}$.

REMARK. (6.6) remains true if $(\partial\xi_2^{\tilde{\phi}}/\partial x_1)$ is replaced by $\partial\xi_1^{\tilde{\phi}}/\partial x_2$ in the left-hand side and $(-1)^q$ by $(-1)^p$ in the right-hand side.

Proof. Let η_2 be the element representing a longitude of the second component L_2 of L in the standard presentation $\mathscr{P}_s(G)$.

Then (6.4) implies that $\eta_2^{\theta}=\xi_2^{\tilde{\phi}}$, where $\theta=\theta_{p+q+1}$ and $\tilde{\phi}=\phi_{p+q+1}$. Thus, in order to prove (6.6) it suffices to show that

$$(6.7)\qquad \frac{1}{p!q!}\,D(x_1^p, x_2^q)^0\left(\frac{\partial \eta_2^{\theta}}{\partial x_1}\right)^{\phi} \equiv (-1)^q\,\frac{1}{p!q!}\,D(x_1^p, x_2^q)^0\,\Delta(x_1, x_2) \bmod A^{(p+q+1)}(L).$$

However, this is essentially what we have proved in [6]. To ensure it we shall follow the proof of Theorem 4.1 in [6]. To avoid a confusion on notation involved, it should be noted that x_1, x_2 in (6.7) are denoted by x, y in [6]. First, for the case $p+q=0$, Theorem 6.1 is true. Now we may assume inductively that the theorem is true for $(r, s)<(p, q)$. From the chain rule, it follows that

$$\frac{\partial\eta^\theta}{\partial x} = \sum\left(\frac{\partial\eta}{\partial x_i}\right)^\theta\left(\frac{\partial x_i^\theta}{\partial x}\right)+\sum\left(\frac{\partial\eta}{\partial y_k}\right)^\theta\frac{\partial y_k^\theta}{\partial x}.$$

Thus we see that

$$\frac{1}{p!q!}D(x^p, y^q)^0\left(\frac{\partial\eta^\theta}{\partial x}\right)^\phi$$

is *equal* to the right-hand side of (6.1) in [6]. On the other hand, (7.1) in [6] holds. By the induction assumption, we obtain

$$\frac{1}{r!}\frac{1}{s!}D(x^r, y^s)^0\,\Delta(x, y) \equiv 0 \bmod A^{(p+q-1)}(L).$$

Thus the same argument as was used in §§7–8 in [6] shows that we need only prove that

$$\text{(6.8)}\qquad \begin{aligned} X_i^\theta(r, s)^0 &\equiv \Gamma_i^{p,q}(p-r, q-s) \bmod A^{(p+q-1)},\\ Y_k^\theta(r, s)^0 &\equiv \Lambda_k^{p,q}(p-r, q-s) \bmod A^{(p+q-1)}. \end{aligned}$$

If in §8 in [6] we replace *modulo* $\Delta^*([p+1, q+1])$ by mod $A^{(p+q-1)}$, the proof given there goes through. Thus (6.6) holds.

7. **A presentation of the Chen group of L.**

In this section we shall find a presentation of $Q(G; q)$ for the link group G.

LEMMA 7.1. *Let $\mathscr{P}_0(G)=(x_i, a_{i,j}:t_{i,j}, t_i)$ be a presentation of the link group G given in §6. Let $R=\langle t_{i,j}, t_i\rangle$. Suppose that $F(q)\cap R=\langle w_1, \dots, w_r, \dots, w_p\rangle$, $w_i\in F$, where $w_1, \dots, w_r$ are linearly independent* mod $F(q+1)$ *and $w_{r+1}, \dots, w_p\in F(q+1)$. Then $F(q+1)\cap R=\langle [w_i, x_j], [w_i, a_{k,l}], w_{r+1}, \dots, w_p, 1\leqq i\leqq r, 1\leqq j, k\leqq n, 2\leqq l\leqq \lambda_k\rangle$.*

Proof. In Lemma A5 [1] put $G=F$, $H=F(q+1)$, $M=F(q)$ and $N=F(q)\cap R$. Then $H\subset M$ and $[M, G]=[F(q), F]\subset F(q+1)$. Then the lemma follows immediately from Lemma A5 in [1].

Consider the following disjoint set $W_{i,j}$ of $F(q)$.

$$\text{(7.1)}\qquad \begin{aligned} W_{1,1} &= \{t_{i,j}\},\\ W_{1,q} &= \{[x_k, t_{i,j}, z_1, \dots, z_{q-2}]\} \quad \text{for } q \geqq 2,\\ W_{2,q} &= \{[t_{i,j}, a_{k,l}, z_1, \dots, z_{q-2}]\} \quad \text{for } q \geqq 2, \text{ and}\\ W_{3,q} &= \{[x_i, x_j, z_1, \dots, z_{k-1}, t_{l,m}, z_{k+1}, \dots, z_{q-2}]\} \quad \text{for } q \geqq 3, \end{aligned}$$

where $z_i\in\mathscr{S}(F)$ and the ϕ-image of any elements in $W_{i,j}$ is standard. Let $W_q=\bigcup_{i=1}^3 W_{i,q}$.

We should note that if g is an insubstantial generator of $F(q)$ then $\tau(g)$ is contained in W_q.

LEMMA 7.2. *The set* W_q *is linearly independent* mod $F(q+1)$.

Proof. Since $t_{ij} \equiv a_{ij}$ mod $F(2)$, W_q is in one-one correspondence to the set of insubstantial generators of $F(q)$. Thus the lemma follows.

Lemma 4.3 implies

$$W_{2,q} \cup W_{3,q} \subset H_\Omega(\infty). \tag{7.2}$$

In the following, we always assume that $n=2$, i.e. L has only two components and we use x, y instead of x_1, x_2; ζ instead of ξ_2.

Now consider $t_2=[\zeta, y]$. Let p be the *first* integer such that $\tau_p(t_2)^\sigma \not\equiv 1$ mod $F(p+1)$. Such an integer p may not exist. If it exists, it is uniquely determined. Thus p will be denoted by $p(G)$. For the sake of convenience, we say $p(G)=\infty$ if p does not exist.

LEMMA 7.3. *Let* $A^{(k)}(L)$ *be the first nonzero member of* $\{A^{(k)}(L)\}$. *Then* $p(G)=k+1$. *Consequently*, $p(G)=\infty$ *iff* $\Delta(x, y)=0$.

A proof follows immediately from Lemma 5.1 and Theorem 6.1.

Now we define a new set R_q in $F(q)$ as follows.

For $q<p(G)$, $R_q=\{\tau_{q+1}(t_2)\}$ and for $q \geqq p(G)=p$,

$$R_q = \{[\tau_p(t_2), x^\lambda, y^\mu], 0 \leqq \lambda, \mu \leqq q-p, \lambda+\mu = q-p\}.$$

LEMMA 7.4. *For* $q \geqq p(G)$, $R_q \cup W_q$ *is linearly independent* mod $F(q+1)$.

Proof. Suppose $\tau_p(t_2)^\sigma \equiv \prod [x, y, x^\gamma, y^\delta]^{\beta(x,y,x^\gamma,y^\delta)}$ mod $F(p+1)$. Since $\tau_p(t_2)^\sigma \not\equiv 1$ mod $F(p+1)$, some β are not zero. Then

$$[\tau_p(t_2), x^\lambda, y^\mu] \equiv \prod_{\gamma,\delta} [x, y, x^{\lambda+\gamma}, y^{\mu+\delta}]^{\beta(x,y,x^\gamma,y^\delta)} \bmod F(q+1)\langle W_q\rangle.$$

This implies the lemma.

LEMMA 7.5. *Let* $g=[W, z]$ *be an element of* $F(q)$, $q \geqq 3$, *where* $W \in W_{q-1}$ *and* $z \in \mathscr{S}(F)$. *Let* $\bar{g}=g^{\phi\rho\tau}$. *Then* $r(g)=g\bar{g}^{-1}$ *is in* $H_\Omega(\infty) \cap F(q+1)$. ρ *denotes the standardization* ρ_q.

Proof. Suppose $W \in W_{1,q-1}$, i.e. $W=[x_k, t_{i,j}, z_1, \ldots, z_{q-3}]$. Then $g=[x_k, t_{i,j}, z_1, \ldots, z_{q-3}, z]$. If $z \geqq x_k$, then from Lemma 3.3 we see that g is congruent to an element

$$g_1 = [x_k, t_{i,j}, z_1, \ldots, z_p, z, z_{p+1}, \ldots, z_{q-3}] \quad \bmod F'',$$

where g_1^ϕ is standard. Since $g^{\phi\rho}=g_1^\phi$, $\bar{g}=g^{\phi\rho\tau}=g_1^{\phi\tau}=g_1$. Thus $r(g)=g\bar{g}^{-1}=gg_1^{-1} \in F''$ $\subset H_\Omega(\infty) \cap F(q+1)$.

If $z<x_k$, then from Lemmas 3.3 and 3.4, it follows easily that $g \equiv g_1^{-1}g_2$ mod F'', where $g_1=[z, x_k, z_1, \ldots, z_p, t_{i,j}, z_{p+1}, \ldots, z_{q-3}]$ and $g_2=[z, t_{i,j}, x_k, z_1, \ldots, z_{q-3}]$.

We should note that g_1^{ϕ}, g_2^{ϕ} are standard and $g_1^{\phi} < g_2^{\phi}$. Let z_m be the first element in Γ occuring in a sequence $z_1, \ldots, z_p, a_{i,j}$. Then, since $\tau(z_m) = t_{r,s} \equiv z_m \bmod F(2)$, it follows from 3.1 that

$$g_1 \equiv g_1' = [z_1, x_k, z_1, \ldots, z_{m-1}, t_{r,s}, z_{m+1}, \ldots, z_p, a_{i,j}, z_{p+1}, \ldots, z_{q-3}] \bmod F(q+1)$$

and hence, $g \equiv g_1'^{-1} g_2 \bmod F(q+1)$. Since $g^{\phi\rho} = (g_1'^{\phi})^{-1} g_2^{\phi}$, $\bar{g} = g^{\phi\rho\tau} = ((g_1'^{\phi})^{-1} g_2^{\phi})^{\tau} = g_1'^{-1} g_2$. Therefore, $r(g) = g\bar{g}^{-1} = g(g_1'^{-1} g_2)^{-1} \in F(q+1)$. Next, we have to show that $r(g) \in H_\Omega(\infty)$. Since g_1' satisfies the assumption in Lemma 4.2, $g_1' \in H_\Omega(\infty)$. Therefore, Lemma 4.3 shows that $g_2'^{\phi\tau} = g_1' \in H_\Omega(\infty)$. Thus, it remains only to show that g_2 and g belong to $H_\Omega(\infty)$.

We consider two cases.

Case 1. Some element in Γ occurs in a sequence $z_1, \ldots, z_{q-3}$.

Then, from Lemma 4.2 we see that g^{ϕ} and g_2^{ϕ} belong to $H_\Omega(\infty)$, and hence, both $g = g^{\phi\tau}$ and $g_2 = g_2^{\phi\tau}$ are in $H_\Omega(\infty)$.

Case 2. $z_i \in \Omega$ for all i.

Then $g_2 = [z, t_{i,j}, x_k, z_1, \ldots, z_{q-3}]$ and $g_1 = [z, x_k, z_1, \ldots, z_{q-3}, t_{i,j}]$. Since $(g_1^{\phi})^{-1} g_2^{\phi} = g^{\phi\rho}$, $\bar{g} = g^{\phi\rho\tau} = (g_1^{\phi\tau})^{-1} g_2^{\phi\tau} = g_1^{-1} g_2$. Therefore, $r(g) = g\bar{g}^{-1} = g(g_1^{-1} g_2)^{-1} \in F'' \subset F(q+1) \cap H_\Omega(\infty)$.

Thus, we have proved that if $W \in W_{1,q-1}$ then $r(g) \in H_\Omega(\infty) \cap F(q+1)$.

In the other case where $W \in W_{2,q-1}$ or $W_{3,q-1}$, the exact same method is available, but the proof is much shorter, because we already know that g and $\bar{g}$ are in $H_\Omega(\infty)$. So we shall omit the details.

LEMMA 7.6. *$R \cap F(q) = \langle W_q, R_q, K_q \rangle$, where K_q is a certain collection of elements in $H_\Omega(\infty) \cap F(q+1)$ and it will be defined in the proof.*

Proof. In the case $q=1$, since $F(1) \cap R = R = \langle t_{i,j}, t_i \rangle$, Lemma 7.6 is certainly true, where K_1 is empty. Suppose that Lemma 7.6 is true for $r < q$. First we assume that $q < p(G)$. Since R_q and K_q are subsets of $F(q+1)$, Lemma 7.1 implies that $F(q+1) \cap R = \langle \tilde{W}_{q+1}, K_q, R_q \rangle$, where $\tilde{W}_{q+1} = \{[w, z], w \in W_q, z \in \mathscr{S}(F)\}$. We note that $\tilde{W}_{q+1} \supset W_{q+1}$. Take an element $g = [w, z]$ from $\tilde{W}_{q+1} - W_{q+1}$. Then by Lemma 7.5, $r(g) = g(g^{\phi\rho\tau})^{-1}$ is in $H_\Omega(\infty) \cap F(q+1)$. Let K'_{q+1} be the totality of $r(g)$ for $g \in \tilde{W}_{q+1} - W_{q+1}$. This will be a part of K_{q+1} sought. Since, for any standard generator f of $F(q+1)$, f^{τ} is in W_{q+1}, we see that $\langle \tilde{W}_{q+1} \rangle = \langle W_{q+1}, K'_{q+1} \rangle$. Next, take an element g from K_q. Then Lemma 4.5 shows that $\tau_{q+2}(g) \in H_\Omega(\infty) \cap F(q+2)$. Let K''_{q+1} denote the totality of $\tau_{q+2}(g)$ for $g \in K_q$. Then it is verified that $\langle W_{q+1}, K_q \rangle = \langle W_{q+1}, K''_{q+1} \rangle$. Let $K_{q+1} = K'_{q+1} \cup K''_{q+1}$. Then $\langle \tilde{W}_{q+1}, K_q \rangle = \langle W_{q+1}, K'_{q+1}, K_q \rangle = \langle W_{q+1}, K_{q+1} \rangle$. Similarly, we can prove that $\langle R_q, W_{q+1} \rangle = \langle R_{q+1}, W_{q+1} \rangle$. Thus $R \cap F(q+1) = \langle W_{q+1}, R_{q+1}, K_{q+1} \rangle$.

Now consider the case where $q \geqq p(G)$. Since $W_q \cup R_q$ is linearly independent $\bmod F(q+1)$, it follows that $F(q+1) \cap R = \langle \tilde{W}_{q+1}, K_q, \tilde{R}_{q+1} \rangle$, where $\tilde{R}_{q+1} = \{[\tau_p(t_2), x^{\lambda}, y^{\mu}, z]\}$. Of course, $\tilde{R}_{q+1} \supset R_{q+1}$. Let K'_{q+1}, K''_{q+1} be the same sets as are defined in the previous paragraphs. Then $\langle \tilde{W}_{q+1}, K_q \rangle = \langle W_{q+1}, K'_{q+1}, K''_{q+1} \rangle$.

We shall define the third set K'''_{q+1}. Take an element g from $\tilde{R}_{q+1}-R_{q+1}$. g is of the form: $[\tau_p(t_2), x^\lambda, y^\mu, z]$. If z is in Γ then $g \in H_\Omega(\infty) \cap F(q+1)$. Thus $s(g)=\tau_{q+2}(g)$ is in $H_\Omega(\infty) \cap F(q+2)$ by Lemma 4.5. If z is in Ω then $z=x$ and $s(g)=g[\tau_p(t_2), x^{\lambda+1}, y^\mu]^{-1}$ lies in F'' and hence, $s(g) \in H_\Omega(\infty) \cap F(q+2)$. Let K'''_{q+1} be the totality of $s(g)$ for $g \in \tilde{R}_{q+1}-R_{q+1}$. Then $\langle R_{q+1}\rangle=\langle R_{q+1}, K'''_{q+1}\rangle$. Let $K_{q+1}=K'_{q+1} \cup K''_q \cup K'''_{q+1}$. Then $\langle \tilde{W}_{q+1}, \tilde{R}_{q+1}, K_q\rangle=\langle W_{q+1}, R_{q+1}, K_{q+1}\rangle$. This proves the lemma.

Lemma 7.7. *If g is an insubstantial generator of $F(q)$, then $gF(q+1)$ is contained in $(F(q) \cap R)F(q+1)$.*

Proof. We know from Lemma 7.6 that $F(q) \cap R=\langle W_q, R_q, K_q\rangle$. Let $g=[z_1, \dots, z_q]$ and z_i the first element of Γ occurring in the sequence $z_1, \dots, z_q$. Then, since $z_i \equiv t_{j,k} \bmod F(2)$, we see that

$$\begin{aligned} gF(q+1) &= [z_1, \dots, z_q]F(q+1) \\ &= [z_1, \dots, z_{i-1}, t_{j,k}, z_{i+1}, \dots, z_q]F(q+1) \\ &\subset W_qF(q+1) \subset (F(q) \cap R)F(q+1), \end{aligned} \qquad \text{Q.E.D.}$$

From Lemmas 7.6 and 7.7, we obtain immediately

Lemma 7.8. *For the link group G,*

$$Q(G;q) = ([x, y, x^\lambda, y^\mu], 0 \leqq \lambda, \mu \leqq q-2, \lambda+\mu = q-2 : R_q^\sigma, F(q+1)^\sigma).$$

Let $\tau_{p(G)}(t_2)^\sigma \equiv \prod [x, y, x^\lambda, y^\mu]^{\beta(\lambda,\mu)} \bmod F(p+1)$. Since not all $\beta(\lambda, \mu)$ are zero, $d=\text{g.c.d.}_{\lambda,\mu}\{\beta(\lambda, \mu)\}$ is not zero. Then from Lemma 7.8, it follows

Lemma 7.9.

$$\begin{aligned} Q(G;q) &\cong Z^{q-1} \quad \textit{for } q < p(G) = p, q \geqq 2, \\ Q(G;p) &\cong Z_d+Z^{p-2}. \end{aligned}$$

8. **Determination of $Q(G;q)$.** To determine $Q(G;q)$ for $q \geqq p(G)+1$, we need the following

Lemma 8.1. *Let*

$$M(m,n) = \begin{bmatrix} a_1 & \cdots & a_n & & & 0 \\ & a_1 & \cdots & a_n & & \\ & & \ddots & & \ddots & \\ 0 & & & a_1 & \cdots & a_n \end{bmatrix}$$

be an $m \times (n+m-1)$ matrix with entries in the integer ring Z. Let $\varepsilon_\lambda(M)$ be the ideal generated by $\lambda \times \lambda$ minors of $M(m,n)$. Then $\varepsilon_\lambda(M)=(d^\lambda)$, where $d=$g.c.d. $\{a_1, \dots, a_n\}$ and $1 \leqq \lambda \leqq m$.

Proof. For $n=1$ or $m=1$, the lemma is obvious. Thus we assume that the lemma is true for $M(\bar{m}, \bar{n})$, $(\bar{m}, \bar{n})<(m, n)$. Consider $\varepsilon_\lambda(M(m, n))$. First we should note that it is sufficient to show that $\varepsilon_\lambda(M)=Z$ if $d=1$. Since $\varepsilon_\lambda(M(m,n))\supset \varepsilon_\lambda(M(m-1,n))=Z$, it follows that for $1\leqq\lambda<m$, $\varepsilon_\lambda(M(m,n))=Z$ and hence, it remains to show that $\varepsilon_m(M(m,n))=Z$. Or equivalently, $m\times m$ minors of $M(m,n)$ generate Z.

Now by the induction assumption, we know that for the matrix

$$N(m-1,n-1) = \begin{bmatrix} 0 & a_2 & \cdots & a_n & & & 0 \\ & 0 & a_2 & \cdots & a_n & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & 0 & & 0 & a_2 & \cdots & a_n \end{bmatrix},$$

$$\varepsilon_\lambda(N) = (d_1^\lambda), \qquad 1 \leqq \lambda < m, \ \cdot d_1 = \text{g.c.d.}\,(a_2,\ldots,a_n).$$

Consider the set $\tilde{M}(j)$ of all $m\times m$ minors of $M(m,n)$ that contains $(1,j)$ entry a_j. $\tilde{M}(j)$ generates an ideal K_j in Z. Then $\varepsilon_m(M)=K_1+\cdots+K_n$.

We may assume that $a_1\neq 0$, otherwise by the induction assumption, we are done. Then $K_1=(a_1)$, since $\varepsilon_{m-1}(M(m-1,n))=Z$. Consider the matrix $\tilde{N}=N(m-1,n-1)$. Then by the induction assumption, $\varepsilon_{m-1}(\tilde{N})=(d_1^{m-1})$. Therefore, $K_2\subset\varepsilon_{m-1}(\tilde{N})+(a_1)=(d_1^{m-1})+(a_1)$. On the other hand, any element $\tilde{D}$ of $\varepsilon_{m-1}(\tilde{N})$ can be written as $\tilde{D}=a_2D+a_1b$ for some $m\times m$ minor D of $\tilde{M}(2)$ and some integer b. Thus $\varepsilon_{m-1}(\tilde{N})\subset K_2+K_1$. Therefore, $K_1+K_2\subset K_1+\varepsilon_{m-1}(\tilde{N})\subset K_1+K_2$.

Since d_1 and a_1 are relatively prime, so are d_1^{m-1} and a_1. Thus $\varepsilon_m(M)\supset K_1+K_2=Z$.

From Lemmas 7.9 and 8.1, it follows that

LEMMA 8.2. *$Q(G;q)\cong Z_d^{q-p}+Z^{p-1}$ for $q\geqq p(G)+1$, where d is the same integer as in Lemma* 7.9.

On the other hand, the integer d can be described in terms of $A^{(k)}(L)$. In fact, Lemma 5.1 and Theorem 6.1 imply that

$$\text{(8.2)} \qquad \begin{aligned} A^{(q)}(L) &= 0 \quad \text{for } q < p-1, \\ A^{(p-1)}(L) &= d. \end{aligned}$$

Thus we have finally:

THEOREM 8.1. *Let L be a link with two components. Let $d=A^{(p-1)}(L)$ be the first nonzero member of $\{A^{(k)}(L)\}$. Then*

$$\begin{aligned} Q(G;q) &\cong Z^{q-1} && \textit{for } q < p, \\ Q(G;q) &\cong Z_d^{q-p}+Z^{p-2} && \textit{for } q \geqq p. \end{aligned}$$

COROLLARY. *$\Delta(x,y)=0$ iff $Q(G;q)\cong Z^{q-1}$ for any $q\geqq 2$.*

As an application, we shall prove the following

THEOREM 8.2. *Let L be a link with two components. Then $\Delta(x,y)=0$ iff each longitude lies in $G(\infty)$.*

Proof. Let ξ and η be longitudes of L. From the remark given in Theorem 6.1, we see that we need only show that ξ lies in $G(\infty)$ iff $\Delta(x, y)=0$. Further, Theorem 6.1 shows that we need only show that

(8.3) $\quad \xi \in G(\infty)$ iff $D(x^\lambda, y^\mu)^0(\partial\xi^{\phi_{k+1}}/\partial y)^\phi = 0, \quad$ for any k, $0 \leqq k = \lambda+\mu$.

Since (8.3) is equivalent to

(8.4) $\quad \xi \in G(n)$ iff $D(x^\lambda, y^\mu)^0(\partial\xi^{\phi_{k+1}}/\partial y)^\phi = 0, \quad$ for any k, $0 \leqq k = \lambda+\mu \leqq n-2$.

we shall prove (8.4).

Now (8.4) is certainly true for $n=2$. In fact, let $\xi \equiv y^\alpha \prod_{i,j} a_{i,j}^{\beta_{i,j}}$ mod $F(2)$. Then $\xi^{\phi_1} \equiv y^\alpha$ mod $F(2)$. Therefore $(\partial\xi^{\phi_1}/\partial y)^0=\alpha=0$ implies $\xi \equiv \prod_{i,j} a_{i,j}^{\beta_{i,j}}$ mod $F(2)$. Since $a_{i,j}=[x_i, u_{ij}]$ in G, we see that $a_{ij}\equiv 1$ mod $G(2)$. Thus $\xi \in G(2)$. Conversely, if $\xi \in G(2)$, then we can write

(8.5) $$\xi \equiv [x, y]^\alpha \prod [x, a_{ij}]^{\beta_{ij}} \prod [y, a_{ij}]^{\gamma_{ij}} \prod [a_{ij}, a_{kl}]^\delta \text{ mod } F(3).$$

Then $\xi^{\phi_1} \equiv [x, y]^\alpha$ mod $F(3)$. Thus, $(\partial\xi^{\phi_1}/\partial y)^0=0$.

Now we assume inductively that (8.4) is true for any $m<n$. Suppose that

$$D(x^r, y^s)^0(\partial\xi^{\phi_{k+1}}/\partial y)^\phi = 0$$

for any k, $0 \leqq k=r+s \leqq n-2$. Then by the induction assumption $\xi \in G(n-1)$. Thus we can write

(8.6) $$\xi \equiv \prod_{\lambda+\mu=n-3} [x, y, x^\lambda, y^\mu]^{\alpha(\lambda,\mu)} \prod [z_1, \ldots, z_{n-1}]^\beta \text{ mod } F(n),$$

where $[z_1, \ldots, z_{n-1}]$ is a standard element in which at least one of z_i is in Γ. Then $[z_1, \ldots, z_{n-1}]^{\phi_{n-1}} \in F(n)$. Thus $\xi^{\phi_{n-1}} \equiv \prod [x, y, x^\lambda, y^\mu]^{\alpha(\lambda,\mu)}$ mod $F(n)$. Then from Lemma 3.8 and our assumption, we see that

$$D(x^{\lambda+1}, y^\mu)^0(\partial\xi^{\phi_{n-1}}/\partial y)^\phi = (\lambda+1)!\mu!\alpha(\lambda, \mu) = 0.$$

Thus $\xi \equiv \prod [z_1, \ldots, z_{n-1}]^\beta$ mod $F(n)$. Since one of z_i is in Γ, it follows that $[z_1, \ldots, z_{n-1}] \equiv 1$ mod $G(n)$. Hence, $\xi \in G(n)$.

Conversely, we assume that $\xi \in G(n)$. Then we can write

$$\xi = \prod_{\lambda+\mu=n-2} [x, y, x^\lambda, y^\mu]^{\beta(\lambda,\mu)} \prod [z_1, \ldots, z_n] \text{ mod } F(n+1)$$

where $[z_1, \ldots, z_n]$ is a standard element in which at least one of z_i is in Γ. Then

$$\xi^{\phi_{n-1}} \equiv \prod_{\lambda,\mu} [x, y, x^\lambda, y^\mu]^{\beta(\lambda,\mu)} \text{ mod } F(n+1),$$

since $[z_1, \ldots, z_n]^{\phi_{n-1}} \in F(n+1)$ for $n \geqq 2$. Thus for $p+q=n-2$, $D(x^p, y^q)^0(\partial\xi^{\phi_{n-1}}/\partial y) =0$. This completes the proof.

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